J. DIFFERENTIAL GEOMETRY
16 (1981) 347–349

FOLIATIONS ON A SURFACE OF CONSTANT CURVATURE AND THE MODIFIED KORTEWEG–DE VRIES EQUATIONS

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Dedicated to Professor Buchin Su on his 80*th birthday*

ABSTRACT. The modified *KdV* equations are characterized as relations between local invariants of certain foliations on a surface of constant Gaussian curvature.

Consider a surface M, endowed with a C^∞-Riemannian metric of constant Gaussian curvature K. Locally let e_1, e_2 be an orthonormal frame field and ω_1, ω_2 be its dual coframe field. Then the latter satisfy the structure equations

$$(1) \qquad d\omega_1 = \omega_{12} \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_{12}, \quad d\omega_{12} = -K\omega_1 \wedge \omega_2,$$

where ω_{12} is the connection form (relative to the frame field). We write

$$(2) \qquad \omega_{12} = p\omega_1 + q\omega_2,$$

p, q being functions on M.

Given on M a foliation by curves. Suppose that both M and the foliation are oriented. At a point $x \in M$ we take e_1 to be tangent to the curve (or leaf) of the foliation through x. Since M is oriented, this determines e_2. The local invariants of the foliation are functions of p, q and their successive covariant derivatives. If the foliation is unoriented, then the local invariants are those which remain invariant under the change $e_1 \to -e_1$.

Under this choice of the frame field the foliation is defined by

$$(3) \qquad \omega_2 = 0,$$

and ω_1 is the element of arc on the leaves. It follows that p is the geodesic curvature of the leaves.

We coordinatize M by the coordinates x, t, such that

$$(4) \qquad \omega_2 = Bdt, \quad \omega_1 = \eta dx + Adt, \quad \omega_{12} = udx + Cdt,$$

Received September 22, 1981. The first author is supported partially by NSF Grant MCS-8023356.

where A, B, C, u are functions of x, t, and η ($\neq 0$) is a constant. Thus the leaves are given by $t =$ const, and ηx and u/η are respectiviely the arc length and the geodesic curvature of the leaves. Substituting (4) into (1), we get

$$A_x = uB, \quad B_x = \eta C - uA, \quad C_x - u_t = -K\eta B. \tag{5}$$

Elimination of B and C gives

$$u_t = \left(\frac{A'_x}{u}\right)_{xx} + (uA')_x + \eta^2 K \frac{A'_x}{u}, \tag{6}$$

where

$$A' = A/\eta. \tag{7}$$

By choosing

$$A' = -K\eta^2 + \frac{1}{2}u^2, \tag{8}$$

we get

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x, \tag{9}$$

which is the modified Korteweg–de Vries (= *MKdV*) equation.

Condition (8) on the foliation can be expressed in terms of the invariants p, q as follows: By (2) and (4) we have

$$u = \eta p, \quad C = Ap + Bq. \tag{10}$$

If we eliminate B, C in the second equation by using (5), it can be written

$$\eta q = \left(\log \frac{A'_x}{u}\right)_x = (\log p_x)_x. \tag{11}$$

Introducing the covariant derivatives of p by

$$dp = p_1\omega_1 + p_2\omega_2, \quad dp_1 = p_{11}\omega_1 + p_{12}\omega_2, \tag{12}$$

we have

$$p_x = p_1\eta, \quad p_{xx} = p_{11}\eta^2. \tag{13}$$

Hence condition (11) can be written

$$q = (\log p_1)_1. \tag{14}$$

A foliation will be called a K-foliation, if (14) is satisfied. We state our result in

Theorem. *The geodesic curvature of the leaves of a K-foliation satisfies, relative to the coordinates x, t described above, an MKdV equation.*

The above argument can be generalized to *MKdV* equations of higher order. The corresponding foliations are characterized by expressing q as a function of $p, p_1, p_{11}, p_{111}, \cdots$.

Is there a similar geometrical interpretation of the *KdV*-equation itself, which is

(15) $$u_t = u_{xxx} + uu_x?$$

We do not have a simple answer to this question. Unlike the *MKdV*-equation, the sign of the last term is immaterial, because it reverses when u is replaced by $-u$. It is therefore of interest to know that by a different foliation and a different coordinate system one can be led to a *MKdV*-equation (9) where the last term has a negative sign.

For this purpose we put

(16) $$\omega_2 = Bdt, \quad \omega_1 = vdx + Edt, \quad \omega_{12} = \lambda dx + Fdt,$$

where λ is a parameter. Substitution into (1) gives

(17) $$F_x = -KvB, \quad B_x = -\lambda E + vF, \quad E_x - v_t = \lambda B.$$

Suppose $K \neq 0$, we get, by eliminating B, E,

(18) $$v_t = \left(\frac{F'_x}{Kv}\right)_{xx} + (vF')_x + \frac{\lambda^2}{Kv}F'_x,$$

where

(19) $$F = F'\lambda.$$

The choice

(20) $$F' = \frac{K}{2}v^2 - \lambda^2$$

reduces (18) into

(21) $$v_t = v_{xxx} + \frac{3}{2}Kv^2v_x.$$

Here the sign of the second term depends on the sign of K.

It can be proved that the choice (20) corresponds to a foliation which is characterized by

(22) $$q = \frac{p_{11}}{p_1} - 3\frac{p_1}{p} = \left(\log\frac{p_1}{p^3}\right)_1.$$

References

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